

AD-A042 372

NUMERICAL COMPUTATION CORP STONY BROOK N Y  
A NUMERICAL ALGORITHM FOR REMOTE SENSING OF OCEAN DENSITY PROFI--ETC(U)  
FEB 77 Y M CHEN, D S TSIEH

N00014-76-C-0804

NL

UNCLASSIFIED

NCC-1

| OF |  
ADA042372



END  
DATE  
FILMED  
8-77  
DDC

REPORT NCC - 1

12  
B.S.

AD A 042372

A NUMERICAL ALGORITHM FOR REMOTE SENSING OF  
OCEAN DENSITY PROFILES BY ACOUSTIC PULSES

Y. M. Chen and D. S. Tsien  
Numerical Computation Corp.  
22 Meadow Drive  
Stony Brook, New York 11790

February 1, 1977

Final Report

Approved for Public Release; Distribution Unlimited

DDC  
RECEIVED  
AUG 2 1977  
C

NU NO.  
DDC FILE COPY

Prepared under  
Office of Naval Research  
Contract N00014-76-C-0804  
NR 187-904

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NCC - 1	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A NUMERICAL ALGORITHM FOR REMOTE SENSING OF OCEAN DENSITY PROFILES BY ACOUSTIC PULSES,		5. TYPE OF REPORT & PERIOD COVERED Final Report June 1, 1976 - Jan. 31, 1977
7. AUTHOR(s) Yung Ming/Chen and Dar Sun/Tsien		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Numerical Computation Corp. 22 Meadow Drive Stony Brook, N. Y. 11790		8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0804
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research, Dept. of Navy 800 North Quincy Street Arlington, Virginia 22217		10. PROGRAM ELEMENT, SUBJECT, TASK AREA & WORK UNIT NUMBERS 62711N RF 11-121-804
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Office of Naval Research, Resident Representative 715 Broadway, 5th Floor New York, N. Y. 10003		12. REPORT DATE February 1, 1977
16. DISTRIBUTION STATEMENT (of this Report) Approved for Public Release; Distribution Unlimited Final rept. 1 Jun 76 - 31 Jan 77		13. NUMBER OF PAGES 32
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) F11121 RF11121804		15. SECURITY CLASS. (of this report) Unclassified
18. SUPPLEMENTARY NOTES To appear in Journal of Computational Physics		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) numerical algorithm, remote sensing, density profiles, acoustic pulses, velocity profiles, inverse problems.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) An iterative algorithm for solving nonlinear inverse problems in remote sensing of ocean density profiles by acoustic pulses is developed. The basic idea of this new algorithm is that first, the original pulse problem in the time-domain is reduced to a continuous wave problem in frequency-domain and then the nonlinear inverse problem in frequency-domain is solved by a hybrid of a Newton-like iterative method, Backus and Gilbert linear inversion technique, and the finite difference method. This new		

SLC 392871.

LB

Block No. 20:

computational algorithm is tested by numerical simulations with given data from ten different frequencies and is found to give excellent results. The effects of taking data from various frequency ranges and of the contaminating instrument and ambient noise on the accuracy and efficiency of numerical computation are investigated. It is found that the low frequency data are preferred over the data from the high frequency spectrum. Under favorable conditions, the maximum pointwise numerical error of the density profile  $\rho(x)$  is less than 5% of the total variation of the density profile,  $p_{tv} \equiv |\text{Max. } \rho(x) - \text{Min. } \rho(x)|$ . Better result can be achieved if larger number of data are available and more efforts are made in the numerical computation. The velocity profile can be obtained from the inferred density profile through a known equation of state.

ADDITION FOR	
NTIS	White Section <input checked="" type="checkbox"/>
DDC	Buff Section <input type="checkbox"/>
UNANNOUNCED	<input type="checkbox"/>
JUSTIFICATION	
BY	
DISTRIBUTION AVAILABILITY CODES	
Dist.	Avail. and/or SPECIAL
A	

A NUMERICAL ALGORITHM FOR REMOTE SENSING OF  
OCEAN DENSITY PROFILES BY ACOUSTIC PULSES

Y. M. Chen and D. S. Tsien

Numerical Computation Corp.

22 Meadow Drive, Stony Brook, New York 11790

SUMMARY

An iterative algorithm for solving nonlinear inverse problems in remote sensing of ocean density profiles by using acoustic pulses is developed. The basic idea of this new algorithm is that first, the original pulse problem in the time-domain is reduced to a continuous wave problem in frequency-domain and then the nonlinear inverse problem in frequency-domain is solved by a hybrid of a Newton-like iterative method, Backus and Gilbert linear inversion technique, and the finite difference method. This new computational algorithm is tested by numerical simulations with given data from ten different frequencies and is found to give excellent results. The effects of taking data from various frequency ranges and of the contaminating instrument and ambient noise on the accuracy and efficiency of numerical computation are investigated. It is found that the low frequency data are preferred over the data from the high frequency spectrum. Under favorable conditions, the maximum pointwise numerical error of the density profile  $\rho(x)$  is less than 5% of the total variation of the density profile, 
$$\rho_{TV} \equiv \left| \max_x \rho(x) - \min_x \rho(x) \right|.$$
 Better result can be achieved if larger number of data are available and more efforts are made in the numerical computation. The velocity profile can be obtained from the inferred density profile through a known equation of state.

## INTRODUCTION

Ocean density profiles can be computed from a finite number of experimental data obtained through acoustic remote sensing techniques as opposed to in situ measurement. Often this type of remote sensing problems can be formulated as "improperly posed" linear or nonlinear inverse problems of partial differential equations in mathematical analysis, and usually the solution of an inverse problem is not unique and does not depend continuously on the given data. To solve linear or nonlinear inverse problems is equivalent to constructing approximate solutions of linear or nonlinear operator equations from inadequate data with or without statistical measurement errors. While the computational methods for solving the traditional direct problems of partial differential equations, e.g., the finite difference method, the finite element method, their hybrids, etc., are quite well developed, the numerical methods for solving inverse problems are still in their infancies. Hence the need of new developments in the computational methods for solving inverse problems is definitely in order.

Recently, several iterative algorithms for solving inverse problems of nonlinear operator equations in Banach spaces have been developed by Chen and Surmont [1],[2], and they have been used for solving a nonlinear radiative transfer problem in the remote sensing of atmospheric temperature profiles [3]. These algorithms are basically hybrids of Newton's iterative methods in Banach spaces [4] and the linear inversion technique of Backus and Gilbert (B & G) [5],[6],[7]. Of course, many other linear inversion techniques, e. g., the regularization method of Tihonov [8], the Moore-Penrose pseudoinverse method [9], etc., can be used just the same. But, the B & G method is preferred

here for the reason that it not only provides an inversion technique, but also can be used as a diagnostic tool for testing the intrinsic resolution of a given set of data for a given problem (or mathematically, provides a sort of pointwise error estimates). More recently, Tsien and Chen [10] has introduced a hybrid of the above-mentioned iterative algorithms and the finite difference method to infer the ocean density profiles by using low frequency acoustic continuous wave (CW) measurements with success; the frequency range is such that the ocean depth is in the range of one to four wavelengths.

Now, a numerical algorithm for solving nonlinear inverse problem in remote sensing of density profiles of a simple ocean model by using acoustic pulses is developed in this paper. Here the adiabatic sound velocity is assumed to be proportional to the inverse square root of the density function. The basic idea of this algorithm is that the incident and reflected acoustic pulses are assumed to be known at the surface of the ocean; through Fourier sine or cosine transform with respect to time  $t$ , the original pulse problem in time-domain is reduced to a CW problem in frequency-domain. Hence the numerical method of [10] can be applied. This new computational algorithm is tested by numerical simulation and it is found to give excellent results. The effects of taking data from various ranges of the frequency spectrum as the input to the numerical algorithm on the accuracy and efficiency of computations are thoroughly studied; the entire frequency range is such that the ocean depth is in the range of 1 - 320 wavelengths. Moreover, the effects of the contaminating instrument and ambient noise on the accuracy of the numerical results or the intrinsic resolution of a given set of measured data are carefully investigated. Finally, a comprehensive discussion

of the numerical results and their implication in actual implementing this computational algorithm is given.

It should be pointed out that there are two basic differences between the new numerical algorithm here and the conventional approach ([11] - [18]) for solving inverse Sturm-Liouville problems. First, a precise knowledge of complete sets of eigenvalues of a particular Sturm-Liouville problem and its associated problems is needed to solve the inverse problem uniquely in the conventional approach, but no such knowledge is needed in the new numerical algorithm here. As a matter of fact, here the choice of the discrete values in frequency-domain  $\{\omega_i\}$ ,  $i = 1, 2, \dots, I$ , (not the square roots of the eigenvalues) is completely arbitrary. Hence they can be chosen for practical and computational conveniences. Secondly, the concept of the uniqueness of solutions which is a major concern in the conventional approach is replaced here by a weaker but more practical one, "the closeness of different numerical solutions to a solution." In the new numerical algorithm here, each iterative calculation with different initial iterate will converge to a different but unique numerical solution [2], nevertheless this non-uniqueness poses no real difficulty for practical problems. If several different numerical solutions are all close enough to the exact solution, then in practical sense any one of them will be an acceptable good solution. Hence in the new numerical algorithm here, the major concern is the closeness of different numerical solutions to the exact solution. In general, the "closeness" is characterized qualitatively by the "spread of the averaging kernel", i. e., the smaller the spread, the closer of the numerical solution to the actual solution.

This is because the "spread" characterizes the "closeness" of the numerical solutions of the individual iterate - linearized problem and the new numerical algorithm here can be shown to be a Newton-like iterative method whose accuracy is in direct proportion to the numerical accuracy in computing the individual iterate [2]. The "spread" is a by-product in the process of computing the numerical solution. The quantitative information on the closeness can be obtained by performing simple numerical simulations.

#### FORMULATION OF PROBLEM

The hydrodynamic structure of the ocean is assumed to be a compressible, inviscous and stratified fluid layer above a rigid plane (Fig. 1). The linearized governing equations in remote sensing of ocean density profiles by using acoustic pulses are [19]

$$\frac{\partial u}{\partial t} = -\rho^{-1} \frac{\partial p}{\partial x}, \quad 0 < x < H, \quad (1)$$

$$\frac{\partial p}{\partial t} = -\rho c^2 \frac{\partial u}{\partial x}, \quad 0 < t,$$

$$\text{with initial conditions} \quad u(x,0) = 0, \quad (2)$$

$$p(x,0) = 0,$$

$$\text{and boundary conditions} \quad u(H,t) = 0, \quad (3)$$

$$p(0,t) = f(t),$$

where  $p$  is the pressure,  $u$  is the particle velocity,  $\rho$  is the equilibrium density,  $c$  is the adiabatic sound velocity related to  $\rho$  through the known equation of state  $c = (\rho K)^{-1/2}$  [20],  $K$  is the compressibility coefficient which is assumed to be constant, and  $f(t)$  is a pulse function with a given duration which corresponds the incident pulse.

The mathematical problem in remote sensing of ocean density profiles is to determine the density function from Eqs. (1),(2),(3) and  $\frac{\partial p}{\partial x}|_{x=0}$ .

To transform the above-mentioned pulse problem in the time-domain to its corresponding CW problem in the frequency-domain, Fourier transform with respect to  $t$  is needed. In particular, due to the homogeneous initial conditions Fourier sine transform is applied to Eqs. (1),(2) and (3).

Upon eliminating the Fourier sine transform of  $u$ , Eqs. (1),(2) and (3) leads to a Sturm-Liouville type of ordinary differential equation for  $\underline{p}(x,\omega)$ , the Fourier sine transform of  $p$ ,

$$\begin{aligned} d/dx (\rho^{-1} d\underline{p}/dx) + \omega^2 K \underline{p} &= 0, \\ \underline{p}(0,\omega) &= \underline{f}(\omega), \\ d\underline{p}(x,\omega)/dx|_{x=H} &= 0, \end{aligned} \quad (4)$$

where  $\underline{f}(\omega)$  is the Fourier sine transform of  $f(t)$  and  $\omega$  is the angular frequency.

Upon normalizing  $K$  to unity and introducing the dimensionless variables,  $x' = x/H$ ,  $\underline{p}' = \underline{p}/\underline{f}(\omega)$ ,  $\rho' = \rho/\rho(0)$ , and  $\omega' = \omega H/c(0)$ , Eq. (4) becomes

$$\begin{aligned} d/dx (\rho^{-1} d\underline{p}/dx) + \omega^2 \underline{p} &= 0, \\ \underline{p}(0,\omega) &= 1, \\ d\underline{p}(x,\omega)/dx|_{x=1} &= 0, \end{aligned} \quad (5)$$

where  $\rho(0)$  and  $c(0)$  are the values of  $\rho(x)$  and  $c(x)$  at  $x = 0$  and they are assumed to be known.

Here the primes are dropped for convenience.

Now, the inverse problem is to determine  $\rho(x)$  from (5) and the known values of  $d\underline{p}(0,\omega)/dx$  for a set of  $\{\omega_i\}$ ,  $i = 1, 2, \dots, I$ . Although Eq. (5) is a linear ordinary differential equation, the inverse problem is nonlinear in nature.

# NUMERICAL ALGORITHM

In general, a nonlinear inverse problem can be solved by using the iterative algorithms of [1] - [3]. For the nonlinear inverse problem here, a variant of the above-mentioned iterative algorithms is used and it is presented in the following.

Let  $p_{n+1} = p_n + \delta p_n$ ,  $\rho_{n+1} = \rho_n + \delta \rho_n$ ,  $n = 0, 1, 2, \dots$  (6) be the  $(n+1)$ th iterates of  $p$  and  $\rho$  such that  $|\delta p_n| < |p_n|$ ,  $|\delta \rho_n| < |\rho_n|$ , and  $\delta \rho_n(0) = 0$ . The  $p_0$  and  $\rho_0$  are the initial guess. Upon substituting (6) into (5) and neglecting terms of  $O\{(\delta p_n)^2\}$ ,  $O\{(\delta \rho_n)^2\}$  and higher, one obtains

$$\begin{aligned} d/dx(\rho_n^{-1} dp_n/dx) + \omega^2 p_n &= 0, \\ p_n(0, \omega) &= 1, \quad dp_n(x, \omega)/dx|_{x=1} = 0, \end{aligned} \quad (7)$$

and

$$\begin{aligned} d/dx(\rho_n^{-1} d\delta p_n/dx) + \omega^2 \delta p_n &= d/dx(\rho_n^{-2} \delta \rho_n dp_n/dx), \\ \delta p_n(0, \omega) &= 0, \quad d\delta p_n(x, \omega)/dx|_{x=1} = 0. \end{aligned} \quad (8)$$

By using the method of Green's function [21], after considerable amount of manipulation one obtains a Fredholm integral equation of the first kind which relates  $\rho_n(x)$  to  $d\delta p_n(x, \omega)/dx|_{x=0}$ ,

$$\int_0^1 \{\rho_n^{-1}(x') dp_n(x', \omega)/dx'\}^2 \delta \rho_n(x') dx' = d\delta p_n(x, \omega)/dx|_{x=0}. \quad (9)$$

Now the inverse problem is reduced to for each iteration the determination of  $\delta \rho_n(x)$  from a set of measured data  $\{d\delta p_i(0, \omega_i)/dx\}$ ,  $i = 1, 2, 3, \dots, I$ . These measured data are used as the input data to the right-hand side of Eq. (9) for the purpose of accelerating the rate of convergence. This inverse problem can be solved by the linear inversion technique of B & G [5] - [7].

It starts with 
$$\delta\rho_n(x) = \sum_{i=1}^I a_{in}(x) d\delta p(0, \omega_i)/dx . \quad (10)$$

Upon substituting Eq. (10) into Eq. (9), one obtains the integral expression,

$$\delta\rho_n(x) = \int_0^1 A_n(x, x') \delta\rho_n(x') dx' , \quad (11)$$

where

$$A_n(x, x') = \sum_{i=1}^I a_{in}(x) \{ \rho_n^{-1}(x') dp_n(x', \omega_i)/dx' \}^2 , \quad (12)$$

is known as the "averaging kernel" and is subjected to the normalization condition,

$$\int_0^1 A_n(x, x') dx' = 1 . \quad (13)$$

The "spread of  $A_n$  from  $x$ " is defined by

$$Q_n(x, A_n) \equiv 12 \int_0^1 (x-x')^2 A_n^2(x, x') dx' . \quad (14)$$

If the measured data are errorless, the set of unknown functions  $\{a_{in}(x)\}$ ,  $i = 1, 2, 3, \dots, I$ , should be chosen such that the averaging kernel  $A_n$  resembles the Dirac delta function  $\delta(x' - x)$  most closely or equivalently, the positive-definite quadratic form of  $\{a_{in}(x)\}$ ,  $i = 1, 2, 3, \dots, I$ ,  $Q_n(x, A_n)$  is minimized subject to the constraint (13). Hence the method of Lagrange multipliers is used.

If the measured data contain errors, then the variance of the density error  $\sigma_n^2(x)$  incurred at  $x$  due to random measurement errors can be found from Eq. (10) to be

$$\sigma_n^2(x) = \tilde{A}_n(x) \cdot \tilde{E} \cdot \tilde{A}_n(x) , \quad (15)$$

where the vector  $\tilde{A}_n(x) \equiv \{a_{in}(x)\}$ ,  $i = 1, 2, 3, \dots, I$ ,  $\tilde{E}$  is the covariance tensor for the random errors of  $\{dp(0, \omega_i)/dx\}$ ,  $i = 1, 2, 3, \dots, I$ , and the means of the random errors are assumed to be zero.

Ideally, one would like to be able to choose  $\tilde{A}_n(x)$  such that both  $\sigma_n^2(x)$  and  $Q_n(x, A_n)$  are minimized. However, this cannot be done, but it is possible to minimize a linear combination of  $\sigma_n^2(x)$  and  $Q_n(x, A_n)$ ,

$$G_n \equiv (1 - s)Q_n(x, A_n) + s \sigma_n^2(x). \quad (16)$$

By varying the parameter  $s$  between zero and unity, the emphasis can be shifted from minimization of the spread to minimization of the error. Thus, there is a tradeoff between intrinsic resolution and the accuracy in the presence of measurement errors. The best choice for  $s$  must be determined by the particular problem.

Now, for a given value of  $s$  the minimization of  $G_n$  subject to the constraint Eq. (13) gives

$$\tilde{A}_n(x) = \tilde{W}_n^{-1}(x) \cdot B_n / B_n \cdot \tilde{W}_n^{-1}(x) \cdot B_n, \quad (17)$$

where the matrix  $\tilde{W}_n(x)$  is defined by

$$\tilde{W}_n(x) = (1 - s)\tilde{Q}_n(x) + s \tilde{E} \quad (18)$$

with the elements of  $\tilde{Q}_n(x)$ ,

$$q_{ijn}(x) = 12 \int_0^1 (x-x')^2 \{ \rho_n^{-1}(x') dp_n(x', \omega_i)/dx' \} \cdot \{ \rho_n^{-1}(x') dp_n(x', \omega_j)/dx' \} dx', \quad (19)$$

$$i, j = 1, 2, 3, \dots, I$$

and the components of  $B_n$ ,

$$b_{in} = \int_0^1 \{ \rho_n^{-1}(x') dp_n(x', \omega_i)/dx' \} dx', \quad i = 1, 2, 3, \dots, I. \quad (20)$$

Finally for each iterate, once  $\tilde{A}_n(x)$  is known from Eq. (17),  $\delta \rho_n(x)$  can be calculated from Eq. (10).

# PROCEDURE FOR NUMERICAL SIMULATION

In order to test the feasibility and to study the general characteristics of the new computational algorithm without the real measurement data, a numerical simulation must be carried out. The procedure is:

- I. First, a typical density profile of the ocean [20]  $\rho^*(x)$  is assumed to be the correct density function. The corresponding pressures  $\{p^*(x, \omega_i)\}$  are obtained from Eq. (5) for a set of angular frequencies  $\{\omega_i\}$ ,  $i = 1, 2, 3, \dots, I$ , by solving the two-point boundary value problem. Then by a finite difference approximation,  $\{dp^*(0, \omega_i)/dx\}$ ,  $i = 1, 2, 3, \dots, I$ , are obtained as the equivalence of the measured data.
- II. The measurement errors can be simulated by multiplying the data,  $\{dp^*(0, \omega_i)/dx\}$ ,  $i = 1, 2, 3, \dots, I$ , by corresponding factors,  $\{(1 + k R_i)\}$ ,  $i = 1, 2, 3, \dots, I$ , where  $k$  is the amplitude factor ( $k = 0.001$  used here) and  $R_i$ 's are the random numbers generated by a random number generator sub-routine [22].
- III. Next,  $\rho_0(x)$  is assumed. Then  $\{p_0(x, \omega_i)\}$  and  $\{dp_0(0, \omega_i)/dx\}$ ,  $i = 1, 2, 3, \dots, I$ , are determined from Eq. (7) by using the same numerical methods as those of Step I. Next, the non-homogeneous term of Eq. (9) can be computed as  $d\delta p_0(0, \omega_i)/dx = (1 + k R_i) dp^*(0, \omega_i)/dx - dp_0(0, \omega_i)/dx$ .
- IV. Finally,  $\delta \rho_0(x)$  is obtained from Eq. (9) and the data  $\{d\delta p_0(0, \omega_i)/dx\}$ ,  $i = 1, 2, 3, \dots, I$ , by using the linear inversion technique of B & G [5] - [7], described in the previous section. Hence from Eq. (6),  $\rho_1(x)$  is obtained.

Other than the truncation, round-off and numerical integration errors, the  $L_2$  norm  $||\rho^*(x) - \rho_n(x)||$  can be used as a criterion for evaluating the

performance of the numerical algorithm consisting of steps III and IV. Although only continuous functions are considered,  $L_2$  norm is used here because of its integrated effect. If the desired accuracy is not met, then one can repeat steps III and IV until the desired accuracy is achieved. The proof of the convergence of this iteration method will be shown elsewhere.

Steps I and III involve solutions of the two-point boundary value problem of the Sturm-Liouville type of ordinary differential equations. In general, a two-point boundary value problem can be solved by using shooting methods [23] with the help of a sophisticated ordinary differential equations solver, GEAR algorithm [24],[25]. However, in the range of  $\omega = 1$  to  $\omega = 30$ , the GEAR algorithm is not as efficient as the simpler finite difference method with second order accuracy, e. g., the center difference approximation. In the range of  $\omega \geq 1,000$  where the oscillation of the solution is extremely high, again the GEAR algorithm becomes inefficient. It is easier to obtain the asymptotic solution of the two-point boundary value problem by using the WKBJ method [26],[27] (Appendix A).

#### NUMERICAL RESULTS WITH ERRORLESS DATA

Numerical simulations are carried out for several  $\rho_0(x)$  with errorless data in the range of angular frequency,  $5 \leq \omega \leq 2,000$ . In performing Step IV, Simpson's rule is used in the numerical integration. The numerical results and their relevant information are given in Table 1 and Figs. 2 - 9. Three typical oscillatory  $\{\rho_n^{-1}(x) d\rho_n(x, \omega_i)/dx\}^2$  are plotted in Fig. 10. The averaging kernels  $A(x_0, x)$  at four different  $x_0$  are shown in Fig. 11.

The primes of the independent and dependent variables are reinstalled in all tables and figures.

Fig.	$\{\omega_i'\}$	$\rho_0'(x')$	n, No. of Iteration	Method Used in Steps I & III	$\ \rho^* - \rho_n'\ ^2$
2	6.98, 8.72, 10.12, 11.86, 13.26, 15.00, 16.40, 18.14, 19.54, 21.28	Smooth, $\ \rho^* - \rho_0'\ ^2 = 15.175$	3	Implicit Finite Difference $\Delta x' = 0.00625$	2.1802
3	Same as Above	Zigzag, $\ \rho^* - \rho_0'\ ^2 = 100.13$	4	Same as Above	8.9106
4	Same as Above	Smooth, $\ \rho^* - \rho_0'\ ^2 = 1,340.4$	2	Same as Above	10.009
5	100, 110, 120, 130, 140, 150, 160, 170, 180, 190	Same as Above	2	Shooting Method, "GEAR"	23.121
6	210, 215, 220, 225, 230, 310, 315, 320, 325, 330	Same as Above	2	Same as Above	25.660
7	410, 415, 420, 425, 430, 435, 440, 445, 450, 455	Same as Above	2	Same as Above	54.434
8	10, 20, 30, 40, 50, 460, 550, 640, 730, 820	Same as Above	2	Same as Above	21.241
9	1100, 1200, 1300, 1400, 1500, 1600, 1700, 1800, 1900, 2000	Same as Above	2	WKB Method	601.92

TABLE 1 . Squares of norms are expressed in units of  $10^{-8}$ .

# NUMERICAL RESULTS WITH ERRONEOUS DATA

In this section, the effects of the contaminating instrument and ambient noise on the accuracy of the numerical results or the intrinsic resolution of a given set of measured data are studied. The measurement errors are simulated by the procedure described in Step II. The covariance tensor of the measurement errors  $\tilde{E}$  is assumed to be a diagonal matrix with elements

$$e_{ii} = \{0.001 R_i dp^*(0, \omega_i)/dx\}^2, \quad i = 1, 2, 3, \dots, I. \quad (21)$$

In consistence with the study of the effects of errorless data from various ranges of the frequency spectrum on the accuracy of the numerical results in the previous section, the example of Fig. 4 is used for the study here. The effect of the weight parameter  $s$  on the accuracy of the numerical result is shown in Fig. 12 and the numerical result  $p_2(x)$  for the case  $s = 0$  is plotted in Fig. 13. The tradeoff curves ( $\sigma_2$  vs.  $s$  curves) for many ocean depth  $x_1 = 0.00625$ ,  $x_2 = 0.10625$ ,  $x_3 = 0.20625$ ,  $x_4 = 0.30625$ ,  $x_5 = 0.40625$ ,  $x_6 = 0.50625$ ,  $x_7 = 0.60625$ ,  $x_8 = 0.70625$ ,  $x_9 = 0.80625$ , and  $x_{10} = 0.90625$  are shown in Fig. 14. Finally, the averaging kernels corresponding to  $s = 0$  at four different ocean depth are plotted in Fig. 15. Again the primes of the independent and dependent variables are reinstalled in all figures.

## DISCUSSION

For the case of errorless data, all of the numerical results in Table 1 and Figs. 2 - 9 indicate that our numerical algorithm has better resolution near the ocean surface than that near the ocean bottom, i. e. the numerical solutions in general are more accurate in the upper region of the ocean than other regions. This conclusion can also be drawn by carefully examining the averaging kernels in Fig. 11 where the averaging kernels resemble the Dirac delta-function most near the ocean surface.

The numerical results in Table 1 and Figs. 2 - 4 suggest that the closer the initial guess  $\rho_0(x)$  to the exact solution  $\rho^*(x)$ , the more accurate the numerical solution will be. However, it can be shown that due to the smallness of the number (inadequate) of data, the limit of the iterates differs from the exact solution and depends on the initial iterate. The detail of this has been discussed in [2]. Hence in the real computation, the initial guess should be made as close to the exact profile as possible.

The numerical results in Table 1 and Figs. 4 - 9 exhibit the fact that data from lower frequency ranges give better numerical accuracy than those from higher frequency ranges. This outcome is not surprising at all, because in the steps I and III of our numerical simulations the ordinary differential equations solvers used here are inefficient in the high frequency range. At the same time, in the step IV of our numerical simulations the kernel of the Fredholm integral equation of the first kind Eq. (9) is much less oscillatory in the lower frequency range than that in the higher frequency range (Fig. 10). This means that the numerical solution of Eq. (9) obtained by using the same size of the integration sub-interval  $\Delta x$  and the same quadrature formula is more accurate in the lower frequency

range than in the higher frequency range; large reduction of the size of  $\Delta x$  for the case of high frequency will not only make the quadrature formula more inefficient but also will greatly increase the globe roundoff error in the numerical computation. Hence in actual computation, low frequency data are much more preferred than those in high frequency range. Moreover, there are several other advantages for using data from the low frequency range and they will be discussed later.

For the case of erroneous data, Fig. 12 shows that the accuracy of the numerical result decreases as the parameter  $s$  increases. Even when  $s = 0$  (Fig. 13) corresponding to the minimization of the spread alone, the accuracy is poorer than the analogous case with errorless data. This means that no matter how small the contaminating instrument and ambient noise can be, they will lower the performance of the numerical algorithm.

The gross behavior of the tradeoff curves (Fig. 14) is the same for all  $x$ . The random error decreases monotonically with increasing spread. The decrease is rapid at small values of the spread, followed by a rather abrupt leveling off with increasing spread. The end point of the curve corresponding to minimum spread and maximum random error corresponds to  $s = 0$  while the opposite end of the curve corresponds to  $s = 1$ . In actual computation, a happy compromising choice of  $s$  is in the range of  $s$  corresponding to the neighborhood of the bottom of the steep portion of the tradeoff curves ( $0.5 \leq (1-s) \cdot 10^7 \leq 1$ ). In this way, the effect of random error is almost minimized, while the spread is also near its minimum. Moreover, upon examining the tradeoff curves for different  $x$ , one finds that in contrary to the case of errorless data where the numerical algorithm has the best resolution near the top of the ocean layer, the random errors in the data have shifted the maximum resolution of the numerical algorithm to somewhere in the middle

of the ocean layer. This fact has also been born out from the study of the averaging kernels for different  $x$  in Fig. 15.

In real situation, there are three major reasons favoring the use of the low frequency data other than the accuracy of the numerical algorithm. First, the low frequency component acoustic waves are less sensitive to the ocean bottom irregularities and other scattering marine objects than the high frequency component waves. For example, if the ocean depth is 5,000 m and the average sound velocity is 1,500 m/sec, then the normalized angular frequency range  $5 \leq \omega' \leq 50$  corresponds to the real frequency range  $0.239 \text{ Hz} \leq f \leq 2.387 \text{ Hz}$  with wavelength  $\lambda$  in the range  $628 \text{ m} \leq \lambda \leq 6,284 \text{ m}$ . Hence it is conservative to say that our numerical algorithm with measured data in the above-mentioned low frequency range will not be sensitive to any irregularity with size less than 500 m.

Secondly, the low frequency component acoustic waves experience less damping due to viscosity when propagate through a fixed distance in the ocean than the high frequency waves, so the received reflected signals will not be drowned out by the contaminating instrument and ambient noise.

Thirdly, low frequency components of both incident and reflected acoustic pulses contain more energy than the high frequency components. Hence in practice, the adverse effects of the contaminating instrument and ambient noise are minimized. To justify the statement about the spectral energy, a numerical simulation of a short acoustic pulse propagating through an ocean layer with the exact density profile  $\rho^*(x)$  is performed in Appendix B. This third advantage of low over high frequencies can be overcome by proper choice of hardware designs, e.g. transducers.

To obtain a more realistic but conservative evaluation of the performance of the numerical algorithm in this paper with data in the low frequency range, the maximum pointwise numerical error should be used as the

evaluation criterion. Upon introducing the total variation of the density profile,  $\rho_{tv} \equiv \left| \text{Max.}_{0 \leq x \leq H} \rho(x) - \text{Min.}_{0 \leq x \leq H} \rho(x) \right|$ , which equals to  $0.00385\rho(0)$  here and is less than  $0.004\rho(0)$  in any real situation [20], our numerical results are given in Table 2.

Fig.	Max. $ \rho^*(x) - \rho_0(x) $ $0 \leq x \leq H$	Max. $ \rho^*(x) - \rho_n(x) $ $0 \leq x \leq H$	Data
2	$0.13 \rho_{tv}$	$0.05 \rho_{tv}$	errorless
3	$0.64 \rho_{tv}$	$0.13 \rho_{tv}$	errorless
4	$1.00 \rho_{tv}$	$0.16 \rho_{tv}$	errorless
13	$1.00 \rho_{tv}$	$0.18 \rho_{tv}$	with random errors

Table 2.

The numerical results indicate that if the low frequency data are errorless and the initial density profile is close enough to the true profile, the maximum possible pointwise numerical error is less than 5% of the total variation of the density profile. The accuracy of the numerical algorithm can be greatly improved if there are more measured data available and improvements in the numerical accuracy are made for each individual step of the numerical algorithm. However, in this way the expense incurred in the numerical computation will be greatly increased. Hence in practice, a priori decision on how to tradeoff must be made for each problem.

#### ACKNOWLEDGMENT

This research has been supported in part by the Office of Naval Research under Contract No. N00014-76-C-0804.

## APPENDIX A

For  $\omega \gg 100$ , the solution of the two-point boundary value problem Eq. (7) becomes highly oscillatory. Hence the GEAR algorithm becomes inefficient. An excellent method to circumvent this difficulty is to construct the asymptotic solution of the two-point boundary value problem for large  $\omega$  by using the WKBJ method [26],[27]. Again in this and next appendixes, the primes are dropped for convenience.

First, upon setting  $v_n(x, \omega) = p_n(x, \omega) / \rho_n^{1/2}(x)$ , Eq. (7) is transformed into

$$\begin{aligned} v_n''(x, \omega) + J_n^2(x, \omega) v_n(x, \omega) &= 0, \\ v_n(0, \omega) &= 1, \\ v_n'(1, \omega) + \frac{1}{2} \rho_n^{-1}(1) \rho_n'(1) v_n(1, \omega) &= 0, \end{aligned} \quad (A1)$$

where  $J_n^2(x, \omega) = \frac{1}{2} \rho_n^{-1}(x) \rho_n''(x) - (3/4)(\rho_n^{-1}(x) \rho_n'(x))^2 + \rho_n(x) \omega^2$  and " ' "  $\equiv d/dx$ . If  $\rho_n(x) \neq 0$  for all  $x$  and  $\rho_n''(x)$  is continuous which is the case here, the asymptotic solution of Eq. (A1) is given by

$$v_n(x, \omega) = J_n^{-1/2}(x, \omega) \{a_n(\omega) \cos \phi_n(x, \omega) + b_n(\omega) \sin \phi_n(x, \omega)\}, \quad (A3)$$

$$\text{where } \phi_n(x, \omega) = \int_0^x J_n(\xi, \omega) d\xi, \quad (A4)$$

$$a_n(\omega) = J_n^{1/2}(0, \omega), \quad (A5)$$

$$\begin{aligned} \text{and } b_n(\omega) &= \{J_n^{1/2}(1, \omega) J_n^{1/2}(0, \omega) \sin \phi_n(1, \omega) + \frac{1}{2} J_n'(1, \omega) J_n^{-3/2}(1, \omega) J_n^{1/2}(0, \omega) \cos \phi_n(1, \omega) \\ &\quad - \rho_n^{-1}(1) \rho_n'(1) J_n^{-1/2}(1, \omega) J_n^{1/2}(0, \omega) \cos \phi_n(1, \omega)\} / \\ &\quad \{J_n^{1/2}(1, \omega) \cos \phi_n(1, \omega) - \frac{1}{2} J_n'(1, \omega) J_n^{-3/2}(1, \omega) \sin \phi_n(1, \omega) \\ &\quad + \frac{1}{2} \rho_n^{-1}(1) \rho_n'(1) J_n^{-1/2}(1, \omega) \sin \phi_n(1, \omega)\}. \end{aligned} \quad (A6)$$

Then  $p_n(x, \omega) = v_n(x, \omega) \rho_n^{1/2}(x)$  is the asymptotic solution of Eq. (7) and it is used in the integrand of Eq. (9).

## APPENDIX B

The propagation of a short acoustic pulse through the ocean layer with the density profile  $\rho^*(x)$  is described by the solution of the following initial boundary value problem of the linear hyperbolic system,

$$\frac{\partial u}{\partial t} = -\rho^{*-1}(x) \frac{\partial p}{\partial x}, \quad 0 < x < 1, \quad (B1)$$

$$\frac{\partial p}{\partial t} = -\frac{\partial u}{\partial x}, \quad 0 < t, \quad (B2)$$

$$\text{with initial conditions } u(x,0) = p(x,0) = 0, \quad (B2)$$

$$\text{and boundary conditions } u(1,t) = 0, \quad (B3)$$

$$p(0,t) = 100 \{1 - \cos(200\pi t)\}, \quad 0 \leq t \leq 0.01$$

$$= 0, \quad t > 0.01.$$

To damp out the stable numerical oscillations (Gibb's phenomenon) in the neighborhood where the solution has either a discontinuity or a steep gradient, an artificial viscosity term  $\nu \frac{\partial^2 u}{\partial x^2}$  is added to the right hand side of the first equation of Eq. (B1)[28]. Along with it, an extra boundary condition,  $\frac{\partial u}{\partial x}|_{x=0} = -20,000\pi \sin(200\pi t)$  for  $0 \leq t \leq 0.01$  and 0 for  $t > 0.01$  must be added to Eq. (B3).

The above system is solved numerically by using a method which is based on the method of lines with the space variable  $x$  discretized by a finite element collocation scheme and the integration of time by the GEAR algorithm [29]. In particular, the Fourier cosine transform of  $u(0,t)$  is computed by using a fast Fourier transform routine [30] and is plotted in Fig. 16. It is obvious that the low frequency spectrum contains more energy than the high frequency spectrum. If the pulse has longer duration, then this phenomenon will be even more pronounced.

## REFERENCES

- [1] Y. M. Chen and J. Surmont, "Iterative Algorithms for Constructing Approximate Solutions of Nonlinear Problems from Erroneous Inadequate Data," *J. Math. Anal. & Appl.*, Vol.46 (1974), 275-288.
- [2] Y. M. Chen and J. Surmont, "Iterative Algorithms for Nonlinear Problems in Remote Sensing," *Applied Math. & Comp.*, Vol.2 (1976), 197-228.
- [3] J. Surmont and Y. M. Chen, "Numerical Solutions of a Nonlinear Radiative Transfer Equation with Inadequate Data," *J. Comp. Phys.*, Vol.13 (1973), 288-302.
- [4] L. V. Kantorovich and G. P. Akilov, Functional Analysis in Normed Spaces, (D. E. Brown, Tr.), Pergamon Press, Oxford, 1964.
- [5] G. Backus and F. Gilbert, "Numerical Applications of a Formalism for Geophysical Inverse Problems," *Geophys. J. Roy. Astrn. Soc.*, Vol.13 (1967), 247-276.
- [6] G. Backus and F. Gilbert, "The Resolving Power of Gross Earth Data," *Geophys. J. Roy. Astron. Soc.*, Vol.16 (1968), 169-205.
- [7] G. Backus and F. Gilbert, "Uniqueness in the Inversion of Inaccurate Gross Earth Data," *Philos. Trans. Royal Soc. Ser. A*, Vol.266 (1970), 123-192.
- [8] A. N. Tihonov, "Solution of Incorrectly Formulated Problems and the Regularization Method," *Soviet Math. Dokl.*, Vol.4 (1963), 1035-1038.
- [9] A. Albert, Regression and the Moore-Penrose Pseudoinverse, Academic Press, New York, 1972.
- [10] D. S. Tsien and Y. M. Chen, "A Numerical Method for Nonlinear Inverse Problems in Fluid Dynamics," Computational Methods in Nonlinear Mechanics, Proc. International Conf. Comp. Methods in Nonlinear Mechs., The University of Texas at Austin, Sept., 1974, pp. 935-943.
- [11] I. M. Gelfand and B. M. Levitan, "On the Determination of a Differential Equation from Its Spectral Function," *Am. Math. Soc. Transl., Series Z*, Vol. 1 (1955), 253-304.
- [12] G. Borg, "Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe," *Acta Math.*, Vol. 78 (1946), 1-96.
- [13] M. G. Krein, "Solution of the Inverse Sturm-Liouville Problem," *Doklady Akad. Nauk SSSR (N.S.)*, Vol. 76 (1951), 21-24.
- [14] H. Hochstadt, "On Inverse Problems Associated with Second-Order Differential Operators," *Acta Math.*, Vol. 119 (1967), 173-192.
- [15] M. L. Gerver and D. A. Kazdan, "On Finding the Function  $\phi(x)$  from the Eigenvalue  $s = s(p)$  of the Equation  $y'' + [\phi(x) - s] y = 0$ ," *Math. U.S.S.R.-Sb.*, Vol. 2 (1967), 199-206.

- [16] O. H. Hald, "On Discrete and Numerical Inverse Sturm-Liouville Problems," Ph. D. Thesis, New York University, New York (1972).
- [17] V. Barcilon, "On the Uniqueness of Inverse Eigenvalue Problems," Geophys. J. R. Astron. Soc., Vol. 38 (1974), 287-298.
- [18] V. Barcilon, "Inverse Problem for a Vibrating Beam," Z. Angew. Math. Phys., Vol. 27 (1976), 347-358.
- [19] L. D. Landau and E. M. Lifshitz, Fluid Mechanics, Pergamon Press, London, 1959.
- [20] G. Neumann and W. J. Pierson, Principles of Physical Oceanography, Prentice-Hall, Englewood Cliffs, New Jersey, 1966.
- [21] I. Stakgold, Boundary Value Problems of Mathematical Physics, Vol. I, MacMillan, New York, 1967.
- [22] The IMSL Library Reference Manual, Ed. 4, International Mathematical and Statistical Libraries, Inc., 1975.
- [23] H. B. Keller, Numerical Methods for Two-Point Boundary-Value Problems, Blaisdell, Waltham, Massachusetts, 1968.
- [24] C. W. Gear, Numerical Initial Value Problems in Ordinary Differential Equations, Prentice-Hall, Englewood Cliffs, New Jersey, 1971.
- [25] A. C. Hindmarsh, "GEAR: Ordinary Differential Equation System Solver," Lawrence Livermore Laboratory Rept. UCID-30001, Rev. 2, 1972.
- [26] P. M. Morse and H. Feshbach, Methods of Theoretical Physics, Part II, McGraw-Hill, New York, 1953.
- [27] G. S. S. Avila and J. B. Keller, "The High-Frequency Asymptotic Field of a Point Source in an Inhomogeneous Medium," Comm. Pure Appl. Math., Vol. XVI (1963), 363-381.
- [28] R. D. Richtmyer and K. W. Morton, Difference Methods for Initial Value Problems, 2nd ed., Interscience, New York, 1967.
- [29] N. K. Madsen and R. F. Sincovec, "General Software for Partial Differential Equations," Recent Developments in Numerical Methods for Differential Systems, ed. by W. E. Schiesser, et. al., Academic Press, New York, 1976.
- [30] R. C. Singleton, "An Algorithm for Computing the Mixed Radix Fast Fourier Transform," IEEE Audio Electroacous., Vol. AU-17 (1969), 83-103.

- Fig. 1 Ocean layer with a rigid bottom.
- Fig. 2 Comparison of the calculated and the exact normalized density profiles, with smooth  $\rho'_0(x')$ , small  $||\rho'^* - \rho'_0||$ , and errorless data in  $6.5 < \omega' < 21.5$ .
- Fig. 3 Comparison of the calculated and the exact normalized density profiles, with zigzag  $\rho'_0(x')$ , small  $||\rho'^* - \rho'_0||$ , and errorless data in  $6.5 < \omega' < 21.5$ .
- Fig. 4 Comparison of the calculated and the exact normalized density profiles, with smooth  $\rho'_0(x')$ , large  $||\rho'^* - \rho'_0||$ , and errorless data in  $6.5 < \omega' < 21.5$ .
- Fig. 5 Comparison of the calculated and the exact normalized density profiles, with smooth  $\rho'_0(x')$ , large  $||\rho'^* - \rho'_0||$ , and errorless data in  $100 \leq \omega' \leq 190$ . \_\_\_\_\_  $\rho'^*$ ; \_\_\_\_\_  $\rho'_0$ ; -----  $\rho'_2$ .
- Fig. 6 Comparison of the calculated and the exact normalized density profiles, with smooth  $\rho'_0(x')$ , large  $||\rho'^* - \rho'_0||$ , and errorless data in  $210 \leq \omega' \leq 330$ . \_\_\_\_\_  $\rho'^*$ ; \_\_\_\_\_  $\rho'_0$ ; -----  $\rho'_2$ .
- Fig. 7 Comparison of the calculated and the exact normalized density profiles, with smooth  $\rho'_0(x')$ , large  $||\rho'^* - \rho'_0||$ , and errorless data in  $410 \leq \omega' \leq 455$ . \_\_\_\_\_  $\rho'^*$ ; \_\_\_\_\_  $\rho'_0$ ; -----  $\rho'_2$ .
- Fig. 8 Comparison of the calculated and the exact normalized density profiles, with smooth  $\rho'_0(x')$ , large  $||\rho'^* - \rho'_0||$ , and errorless data in  $10 \leq \omega' \leq 820$ . \_\_\_\_\_  $\rho'^*$ ; \_\_\_\_\_  $\rho'_0$ ; -----  $\rho'_2$ .
- Fig. 9 Comparison of the calculated and the exact normalized density profiles, with smooth  $\rho'_0(x')$ , large  $||\rho'^* - \rho'_0||$ , and errorless data in  $1100 \leq \omega' \leq 2000$ . \_\_\_\_\_  $\rho'^*$ ; \_\_\_\_\_  $\rho'_0$ ; -----  $\rho'_2$ .
- Fig. 10 Typical  $\{\rho_n'^{-1}(x') dp_n'(x', \omega')/dx'\}^2$  as function of  $x'$  shown for three different values of  $\omega'$ .
- Fig. 11 Typical averaging kernel  $A(x'_0, x')$  as function of  $x'$  shown for four different  $x'_0$  with errorless data in  $6.5 < \omega' < 21.5$ .
- Fig. 12 With erroneous data in  $6.5 < \omega' < 21.5$ , the numerical error of the example in Fig. 4, characterized by  $||\rho'^* - \rho'_2||^2$ , shown to be a monotonic increasing function of  $s$ .
- Fig. 13 Comparison of the calculated and the exact normalized density profiles of the example in Fig. 4 with erroneous data in  $6.5 < \omega' < 21.5$  and  $s = 0$ .

Fig. 14 With erroneous data in  $6.5 < \omega' < 21.5$ , tradeoff curves 1, 2, 3, ..., and 10 corresponding to  $x_0' = 0.00625, 0.10625, 0.20625, 0.30625, 0.40625, 0.50625, 0.60625, 0.70625, 0.80625, \text{ and } 0.90625$  respectively shown. The top and bottom end points correspond to  $s = 0$  and  $s = 1$ , respectively.  $x$  denotes  $s = 1 - 10^{-7}$  and  $\odot$  denotes  $s = 1 - 0.5 \times 10^{-7}$ .

Fig. 15 Typical averaging kernel as function of  $x'$  shown for four different  $x_0'$  with erroneous data in  $6.5 < \omega' < 21.5$  and  $s = 0$ .

Fig. 16 Fourier cosine transform of  $u(0,t)$  plotted as function of  $\omega'$ .

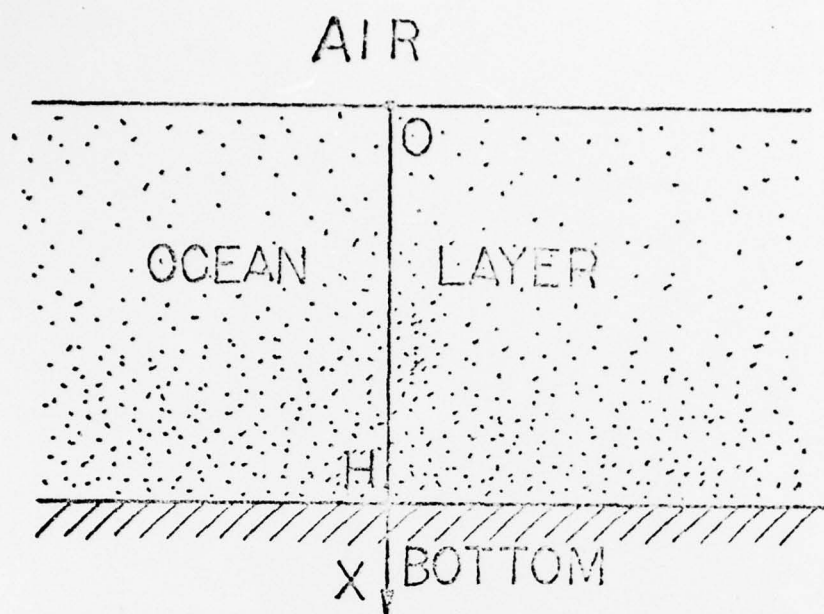


FIG. 1

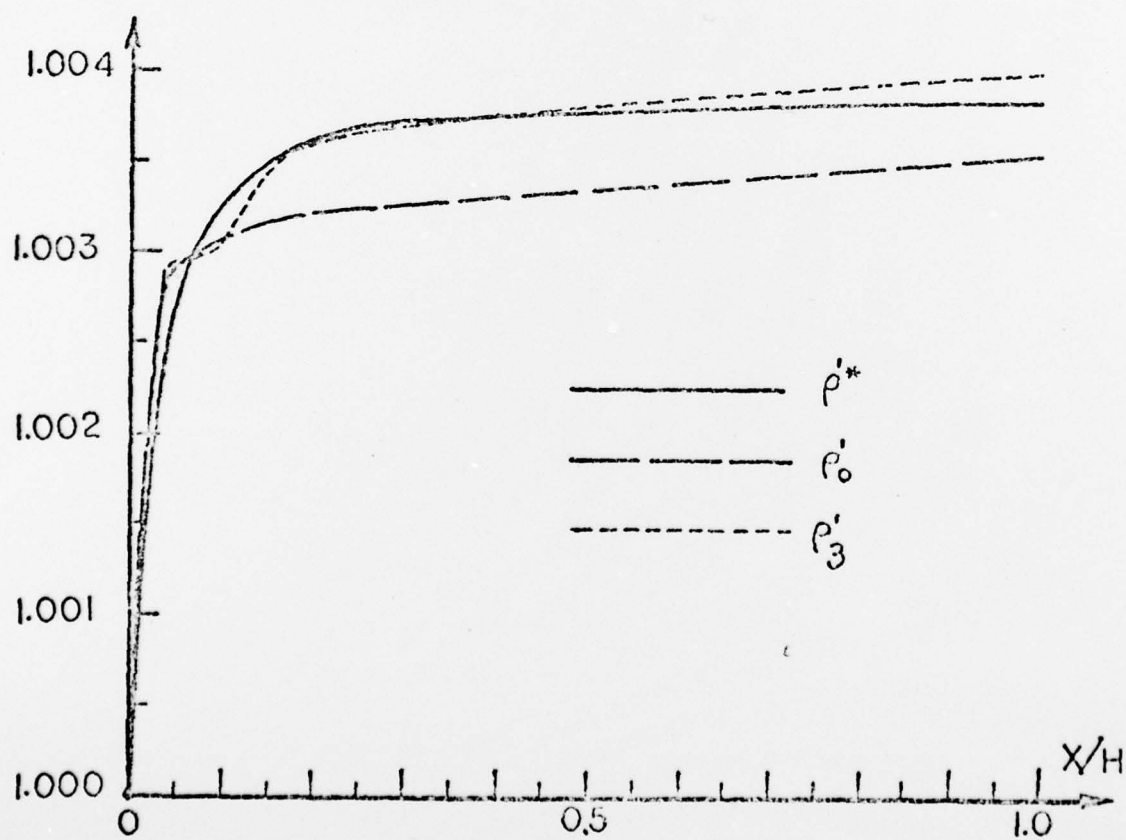


FIG. 2

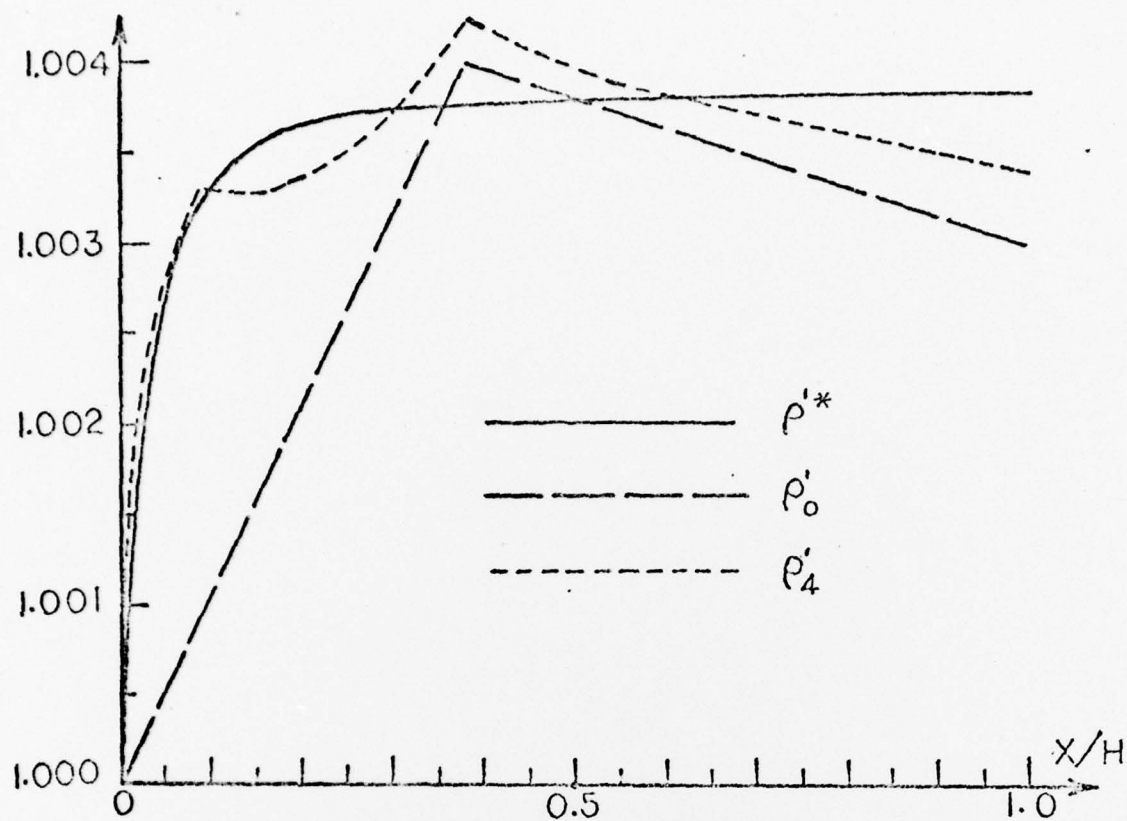


FIG. 3

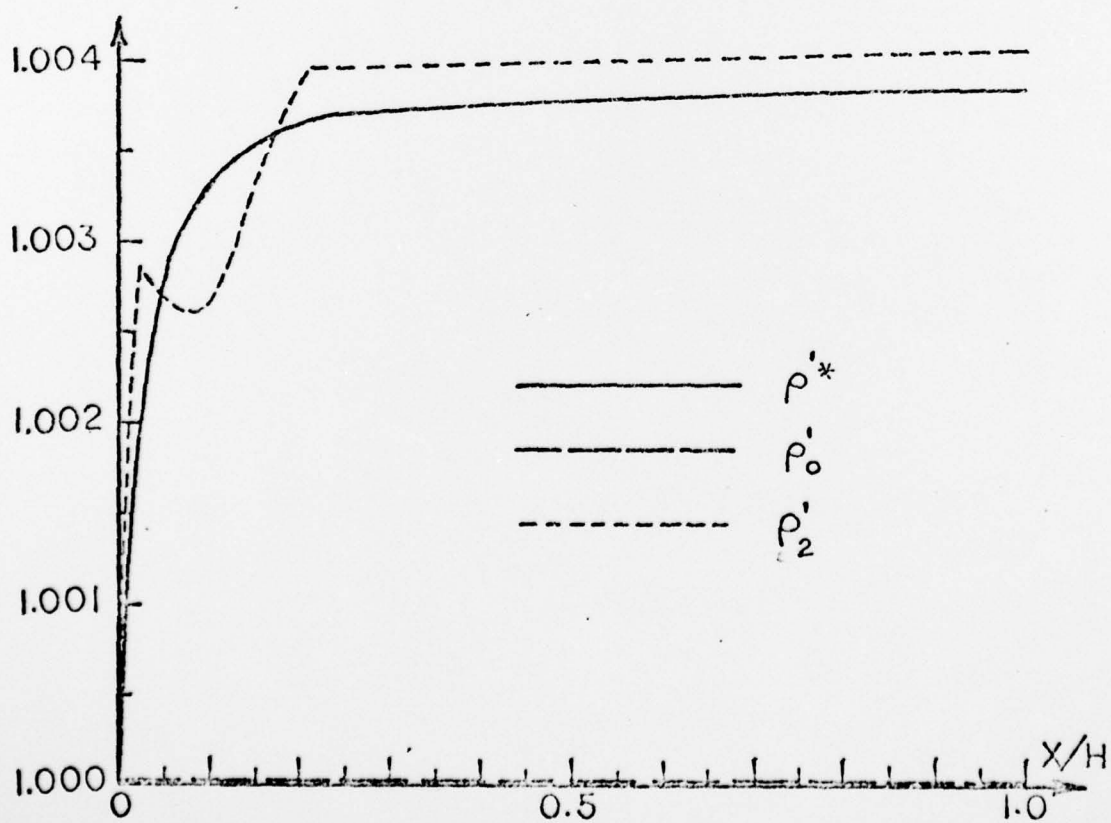


FIG. 4

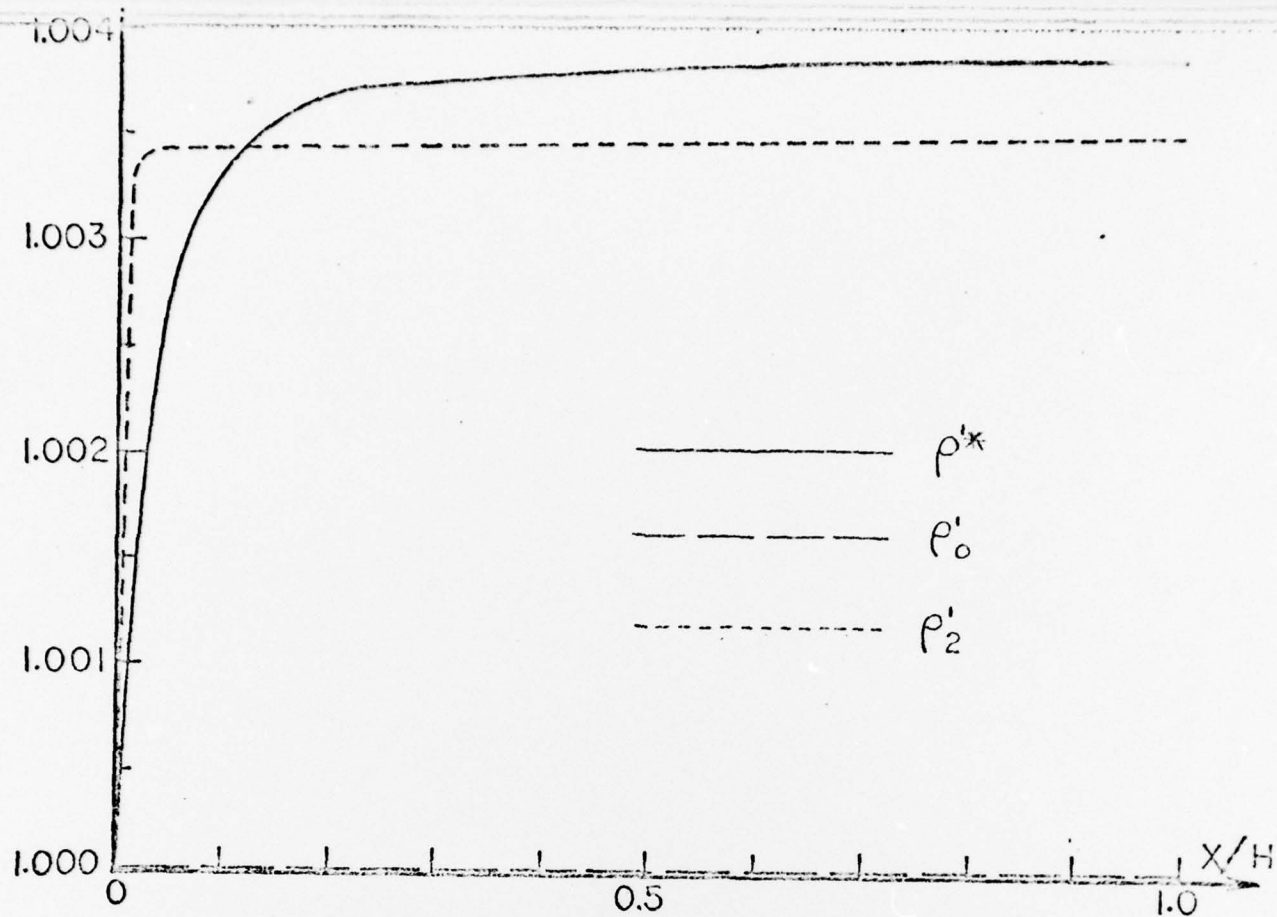


FIG. 5

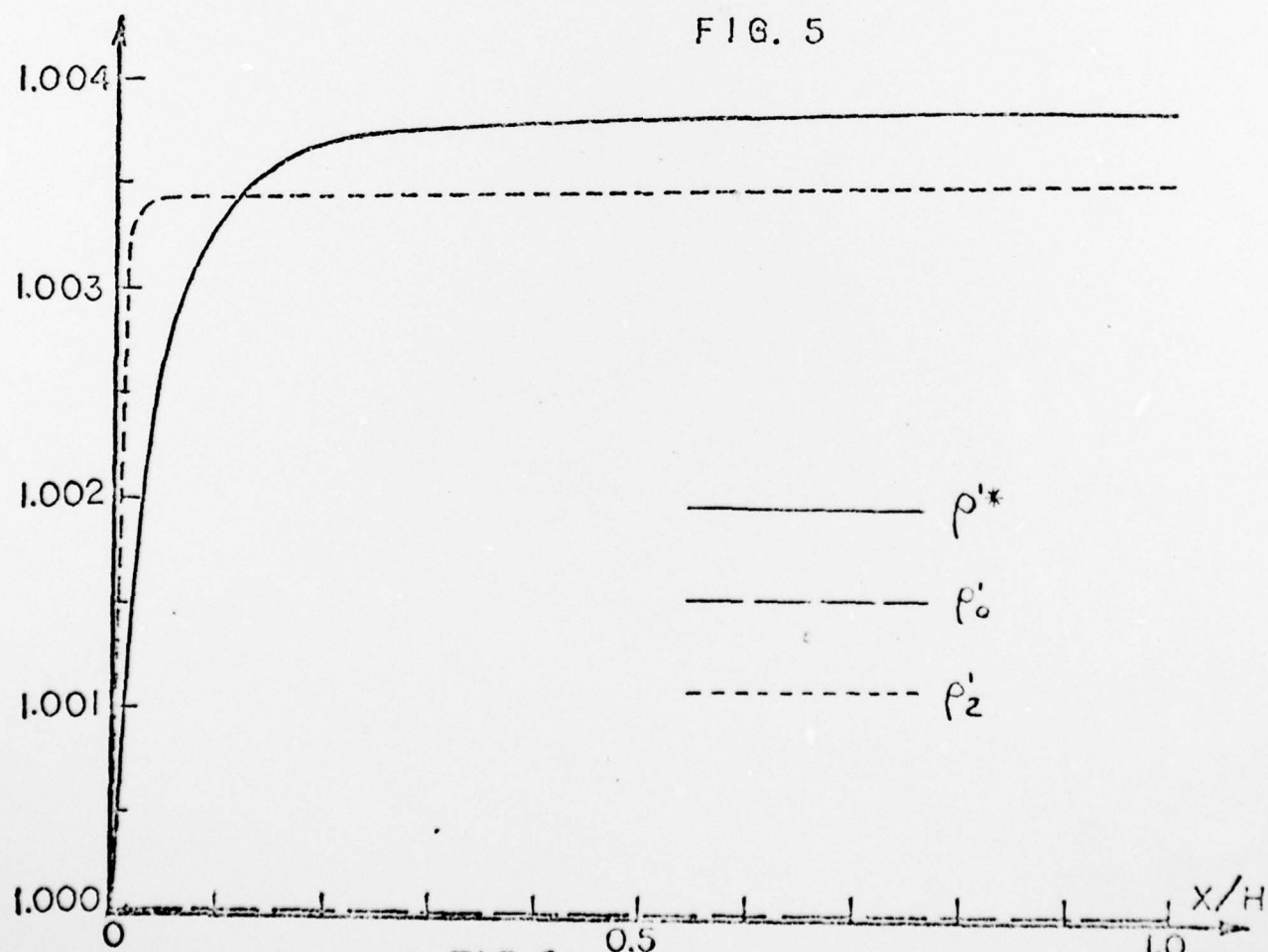


FIG. 6

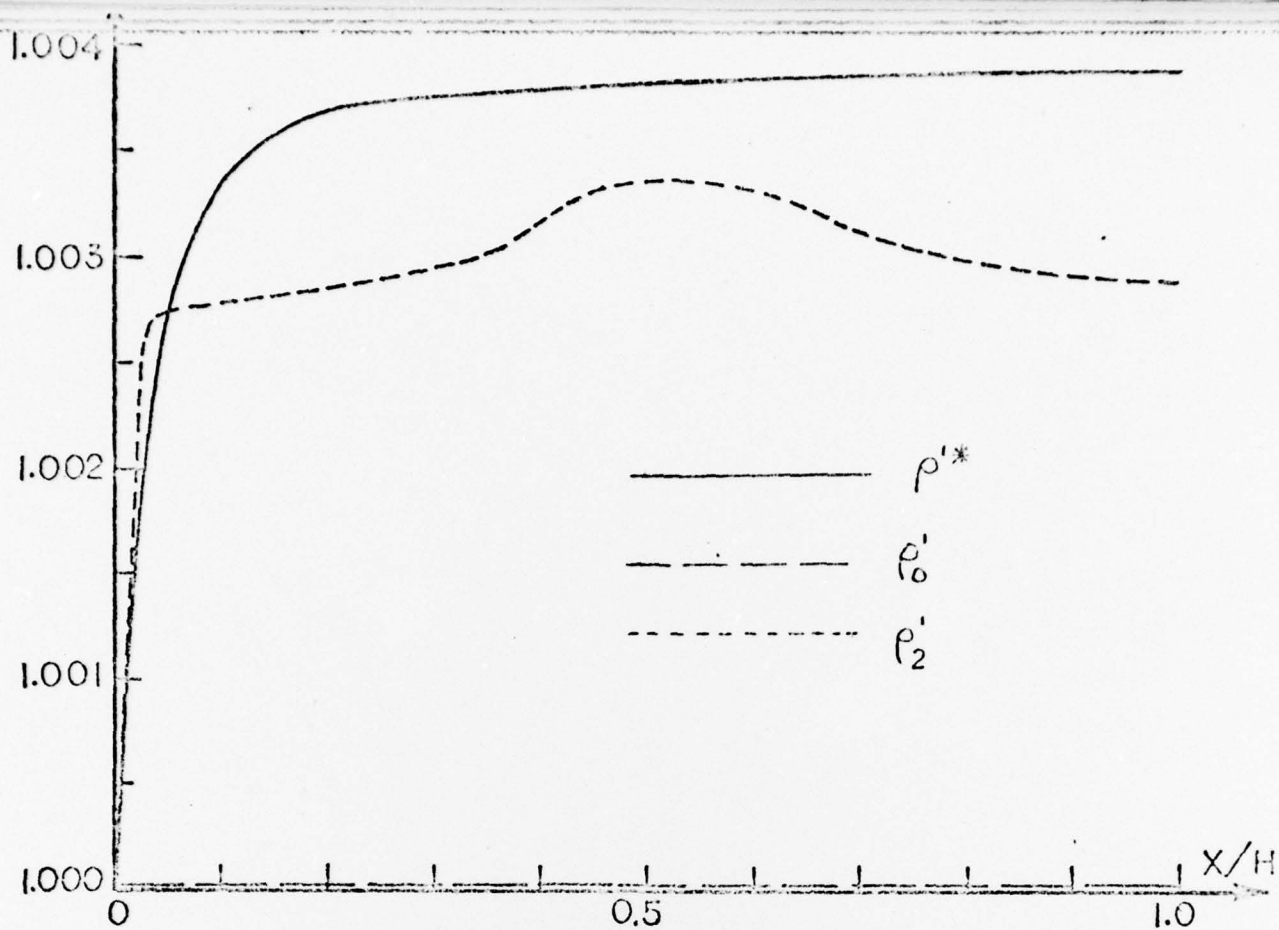


FIG. 7

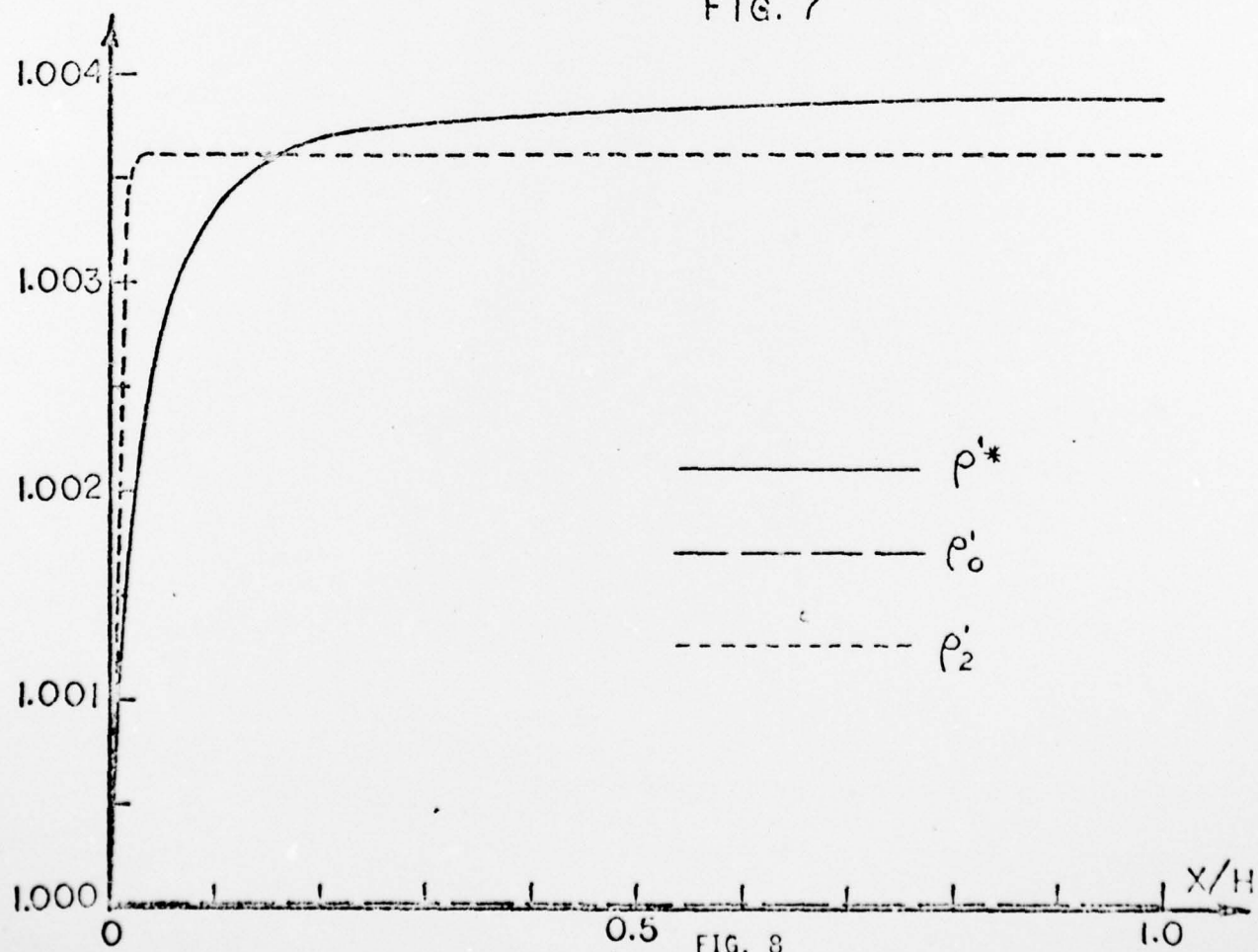


FIG. 8

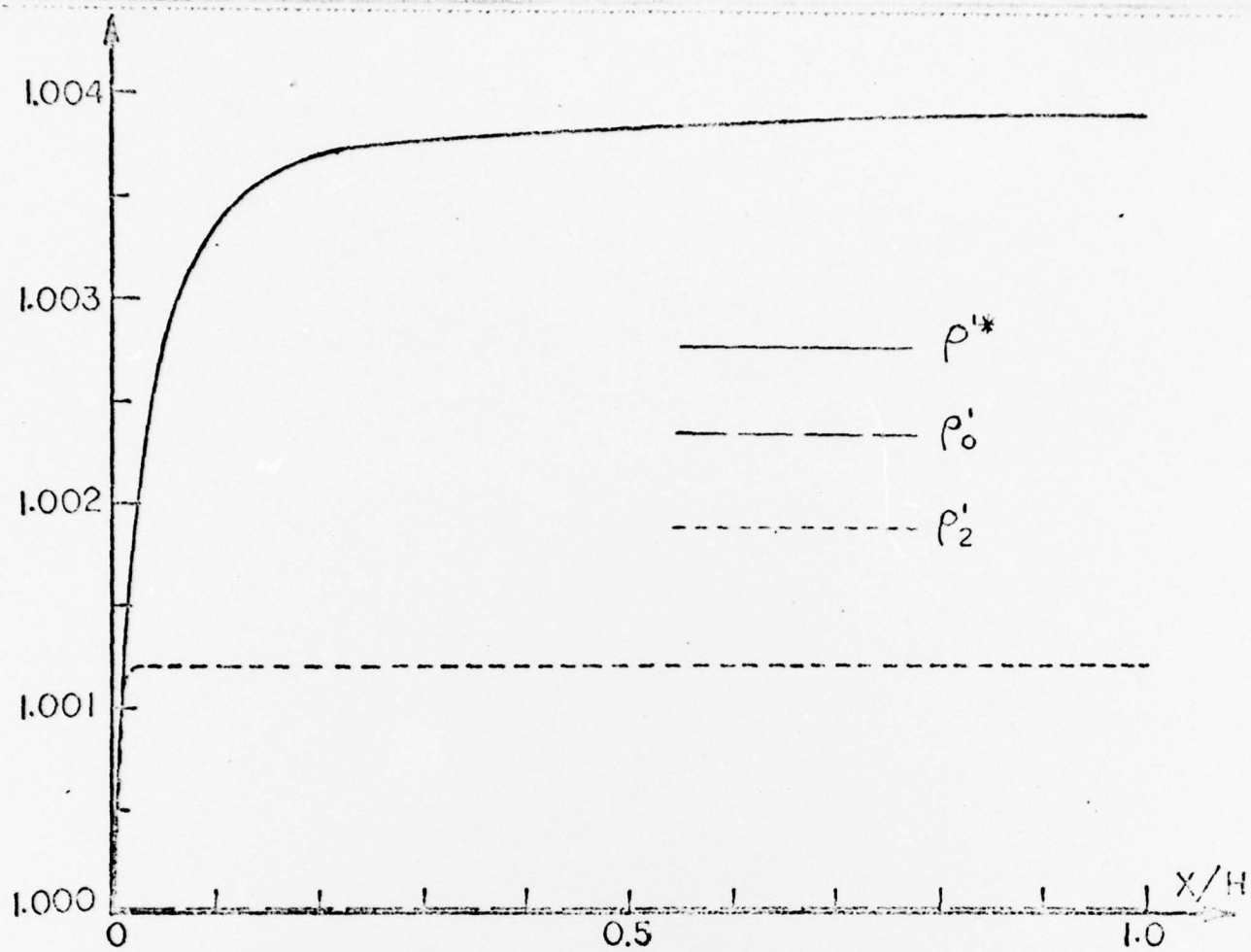


FIG. 9

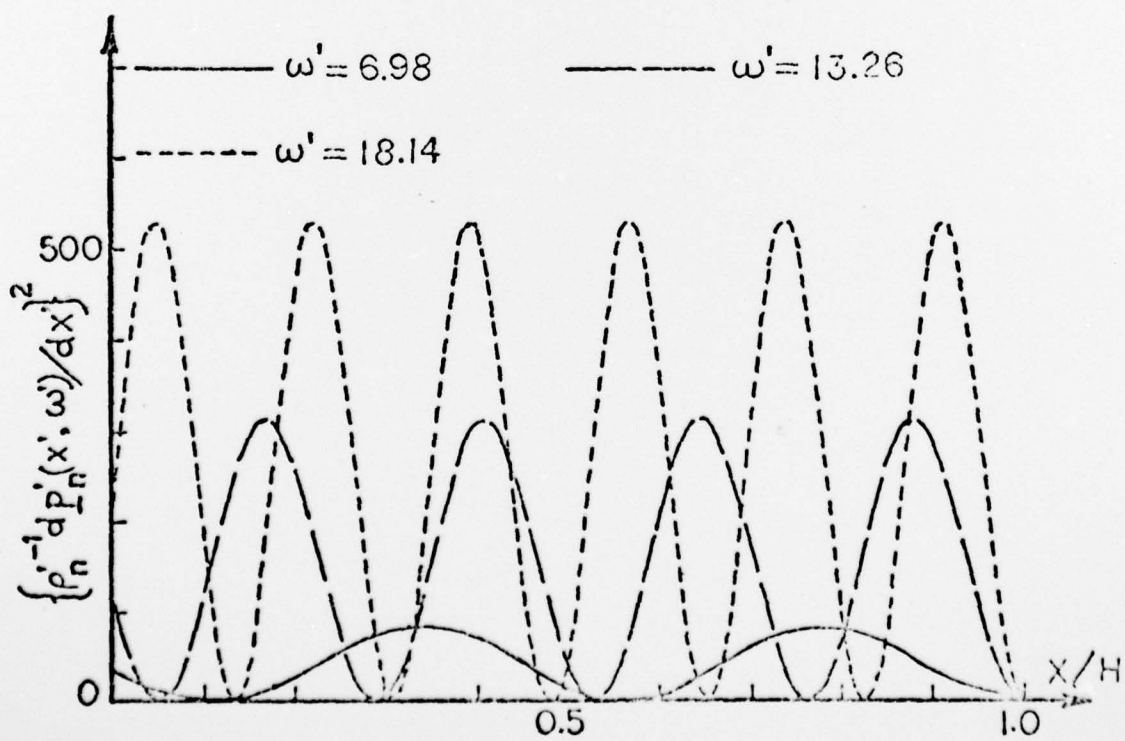


FIG. 10

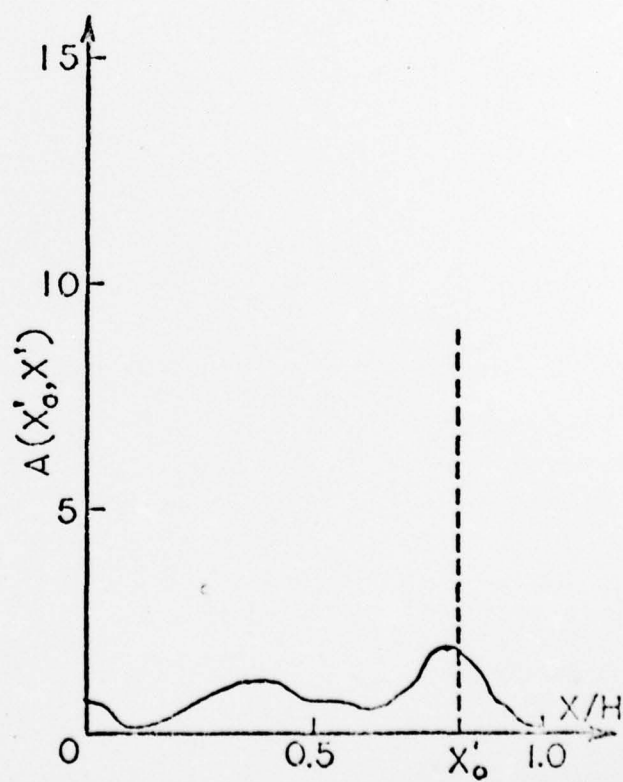
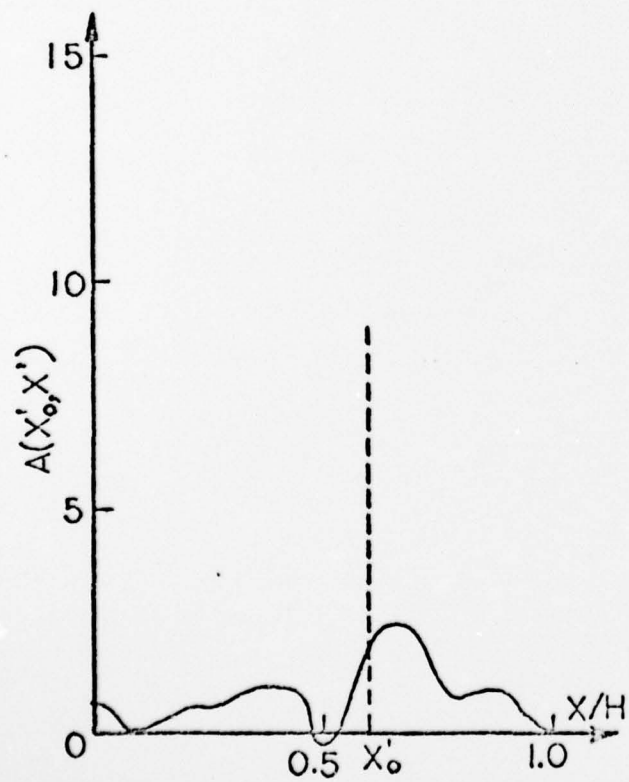
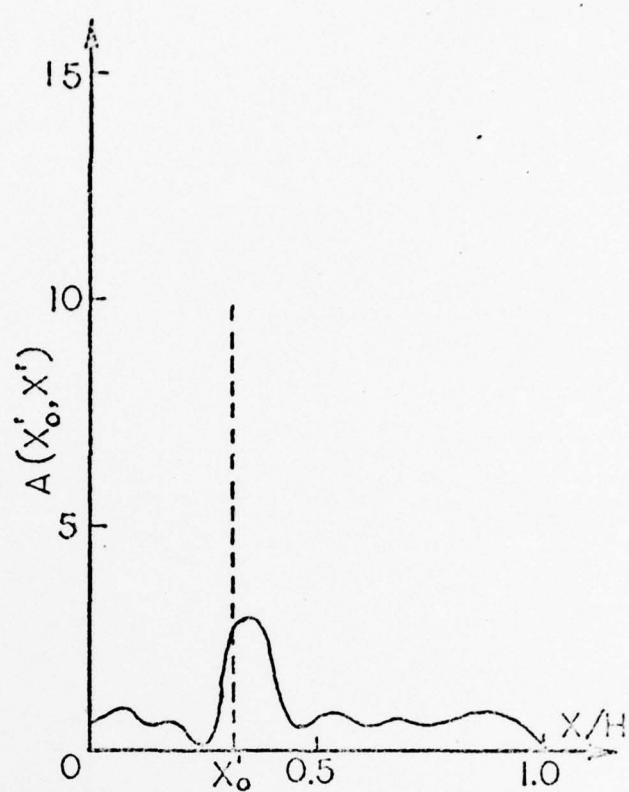
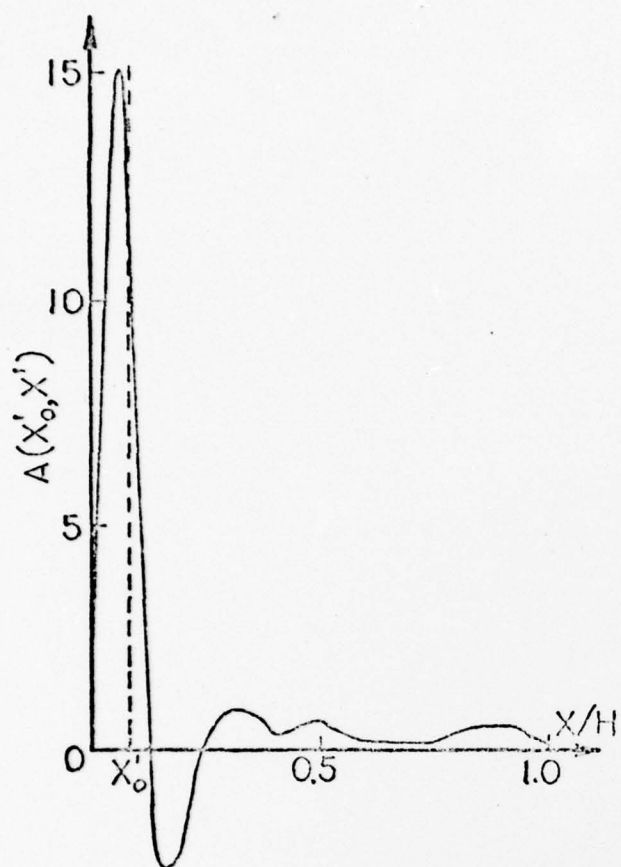


FIG. 11

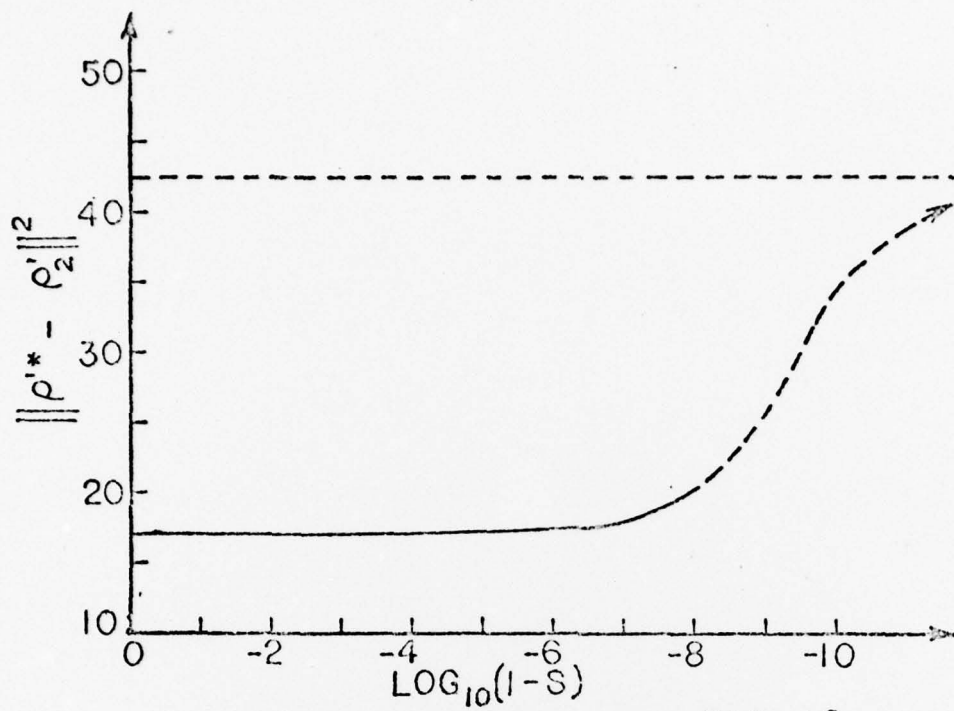


FIG. 12

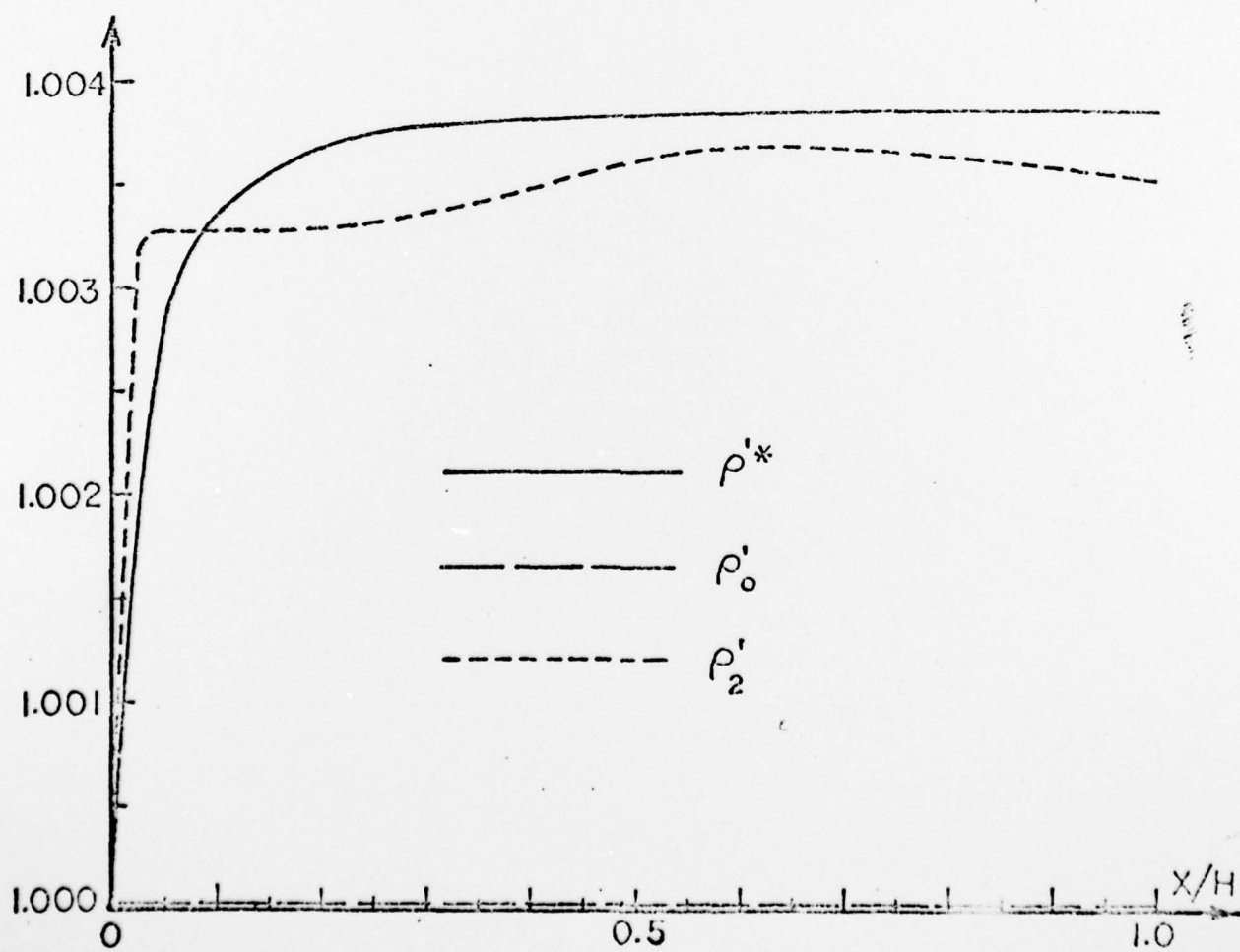
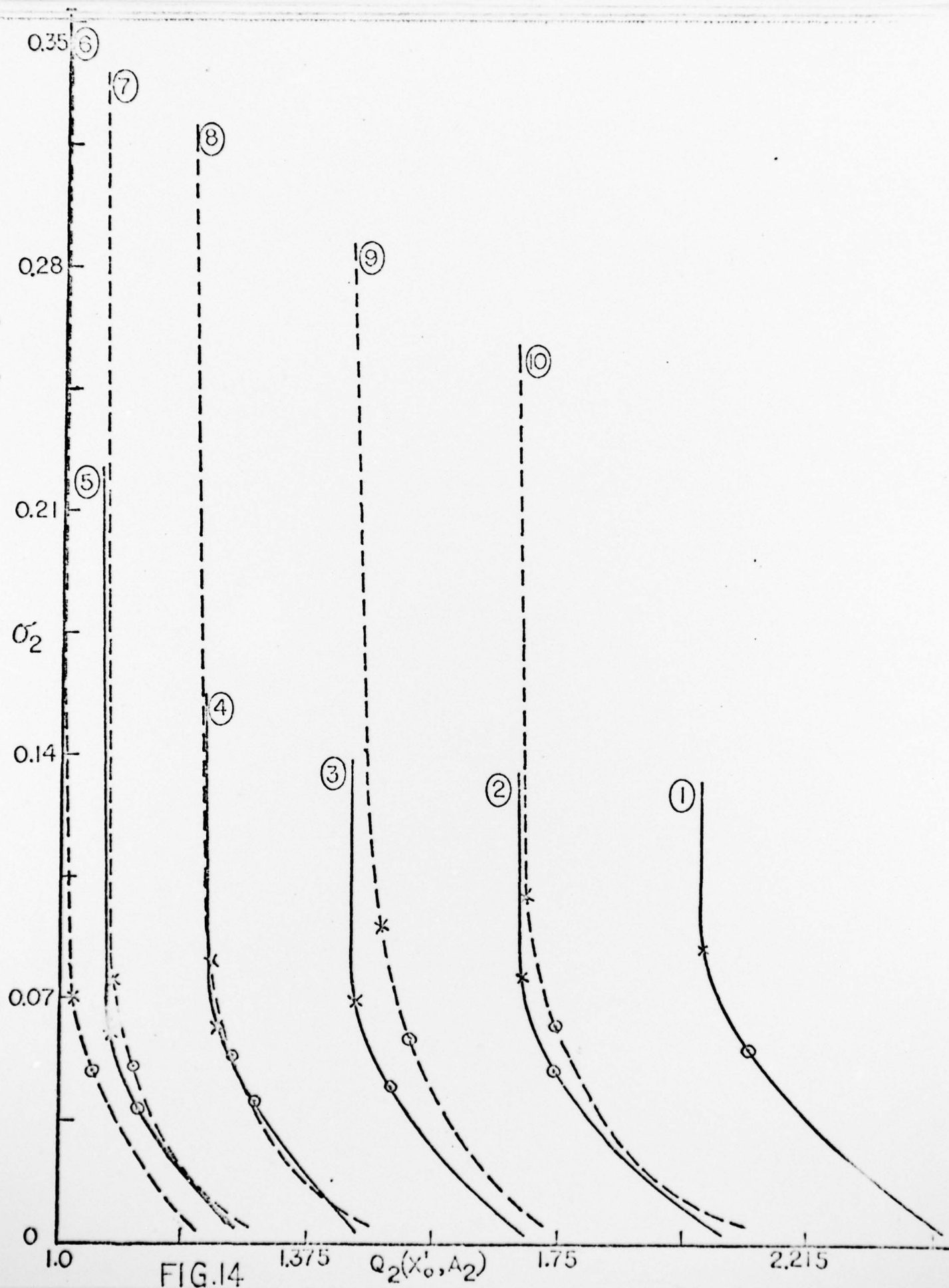


FIG. 13



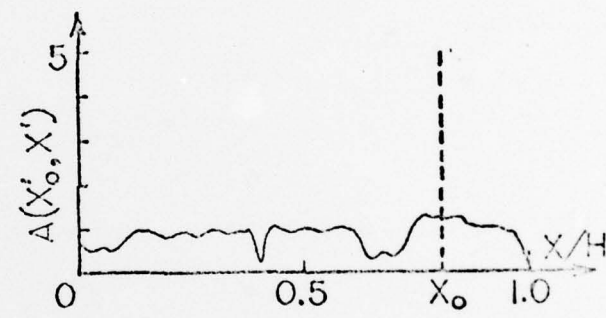
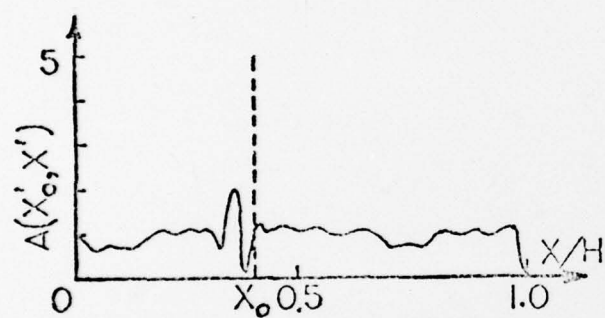
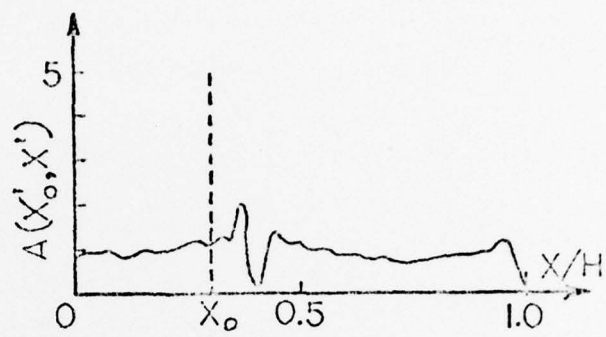
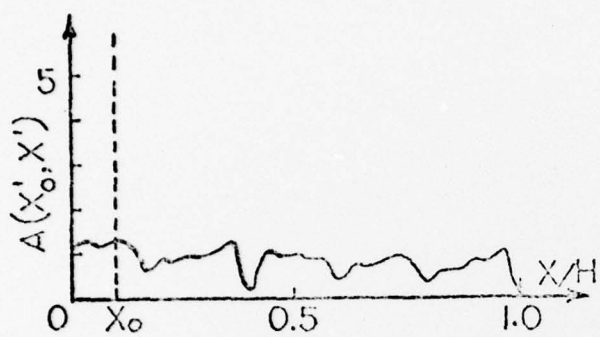


FIG. 15

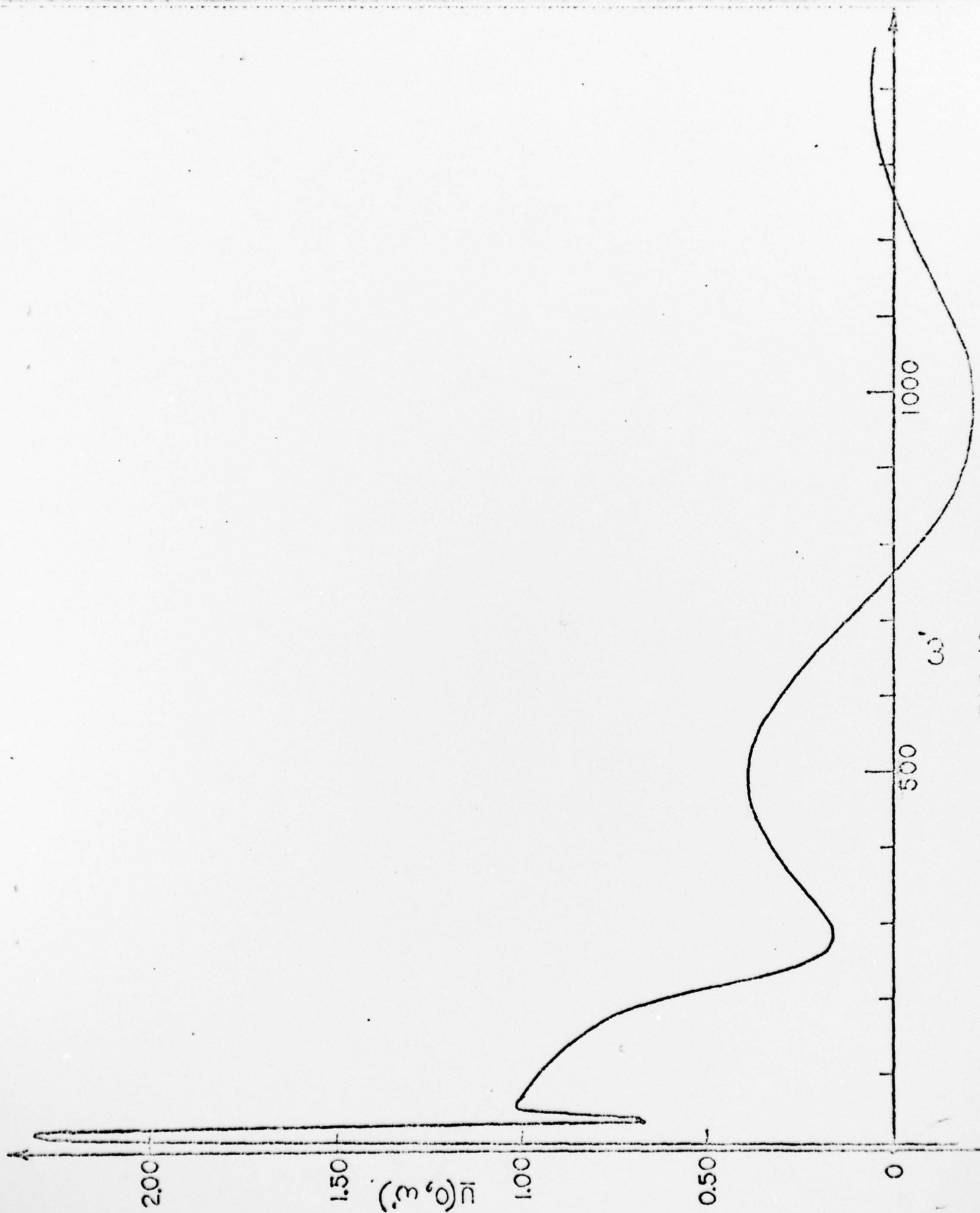


FIG. 16